TEMPERATURE AND VELOCITY DISTRIBUTIONS IN A LAMINAR FLOW OF A FLUID BETWEEN ROTATING COAXIAL CYLINDERS

(TEMPERATURNOYE I SKOROSTNOYE POLIA PRI LAMINARNOM DVIZHENII ZHIDKOSTI MEZHDU VRASHCHAIUSHCHIMISIA KOAKSIAL'NYMI TSILINDRAMI)

PMM Vol.23, No.4, 1959, pp. 740-748

A.I. BORISENKO and A.D. MYSHKIS
(Kharkov)

(Received 12 March 1959)

In most electrical machines the gap between the rotor and the stator is filled with air. Investigations (for example, [1]) have shown that in this case laminar motion is preserved as long as the Reynold's number, $N_{\mathrm{Re}} = \rho W_1(R_2 - R_1)/\mu$, remains less than the critical value, which is equal to $41.2\sqrt{R_m/(R_2 - R_1)}$, where R_2 and R_1 are the radii of the outer and the inner cylinder respectively, μ - the coefficient of viscosity, ρ - the density, W_1 - the linear velocity of the rotating inner cylinder (with a stationary outer one), $R_m = 1/2(R_1 + R_2)$. Numerical computations show that for many existing electrical machines this condition is satisfied, and the motion in the air gap is laminar.

Since a substantial portion of the heat produced in the rotor is transmitted across the air gap, it is important for heat transfer calculations to investigate the state of the air. The use of high voltage insulation, which allows large temperature differences to exist, brings up the question of the effect on the character of the motion of the changes in the physical properties (viscosity and thermal conductivity) of the air due to temperature changes.

In the last few years the electrical machine industry has become familiar with a new type of machine - the so called immersed machine, in which the gap between the rotor and the stator is filled with oil. As is well known, the viscosity of liquids varies considerably with temperature, and consequently the problem posed here is an important one. A similar problem arises, for example, in measuring viscosity by the rotating cylinder method [2], and in other fields of technology.

An investigation of the question when the viscosity and the thermal conductivity are proportional to T^n, with $n = 1$ and $1/2$, is contained in the work of Stepaniants [3]. Closely related questions are examined in the works of Targ ([4], especially Section 25) and Borisenko [5].

The authors wish to express their gratitude to L.G. Loitsiansky, who became acquainted with the work while reviewing it, and made some remarks which were taken into accountin the final editing of the text.

1. Derivation of the field equations and the formulation of the problem. Let us examine a two dimensional ($W_z = 0$, $\partial/\partial z = 0$) laminar uniform ($\partial/\partial t = 0$) axisymmetric ($\partial/\partial \phi = 0$) flow of a liquid or a gas between two coaxial cylinders with radii R_1 and R_2 ($R_1 < R_2$), with the cylinder axis coinciding with the z-axis. Then, from an examination of the portion of the flow enclosed between the cylinders with radii R_1 and R, $R_1 < R < R_2$ (with the same axis), it follows immediately that $W_R = 0$, i.e. the motion of the medium takes place along concentric circles R = const.

It is easy to derive the basic flow equations by examining an elementary layer contained between the cylinders with radii R and $R + dR$; in doing this we will neglect inertia forces. Let ρ denote the density of the medium, and p - the pressure. Since the centrifugal forces must be balanced by the differences of the pressure on the surfaces of this layer,

$$\frac{W^2}{R} 2\pi R\, dR\rho = 2\pi R\, dp, \quad \text{or} \quad \frac{1}{\rho}\frac{dp}{dR} = \frac{W^2}{R} \tag{1.1}$$

The resultant moment of the viscous forces acting on the layer must vanish. Therefore,

$$d\,(p_{R\varphi}\, 2\pi RR) \equiv d\left[\mu\left(\frac{dW}{dR} - \frac{W}{R}\right) 2\pi R^2\right] = 0 \tag{1.2}$$

The expression for $p_{R\phi}$ is easily derived (viz. [4], p. 41, for example). Integrating (1.2) we obtain

$$\mu \frac{d}{dR}\left(\frac{W}{R}\right) + \frac{A}{R^3} = 0 \qquad (A = \text{const}) \tag{1.3}$$

Finally, let us write down the energy equation for the cylindrical layer:

$$d\left(\lambda \frac{dT}{dR} 2\pi R\right) = -E 2\pi R dR \equiv -\mu\left(\frac{dW}{dR} - \frac{W}{R}\right)^2 2\pi R\, dR \tag{1.4}$$

Here E is the dissipation function, the values of which can be obtained, for example, from [4] (p. 48), and λ is the thermal conductivity of the medium, expressed in units of mechanical energy. From (1.3) and (1.4) we get

$$\frac{d}{dR}\left(\lambda R \frac{dT}{dR}\right) + \frac{A^2}{\mu R^3} = 0 \tag{1.5}$$

The constant A and the three integration constants for equations (1.3) and (1.5) must be determined from the four boundary conditions. Two of these are given by prescribing the cylinder speeds, i.e. by prescribing

$$W(R_1) = W_1, \qquad W(R_2) = W_2 \tag{1.6}$$

The other two conditions specify the temperature regime at the surfaces of the cylinders; they consist in prescribing the cylinder temperatures (boundary conditions of the first kind)

$$T(R_1) = T_1, \qquad T(R_2) = T_2 \tag{1.7}$$

or in prescribing the heat flux through the cylinders (boundary conditions of the second kind), or, finally, in prescribing the heat exchange between the cylinders and the medium (conditions of the third kind).

Assume for definiteness that $W_z = 0$, $T_1 > T_2$, and boundary conditions of the first kind (1.7), although all the other cases could be examined in a similar manner. After solving the system of equations (1.3) and (1.5) with boundary conditions (1.6), (1.7), we can determine the pressure, $p(R)$, from equation (1.1) (taking into account the equation of state).

From equation (1.5) it can be seen that every stationary point of the function $T(R)$ is a maximum point; therefore, if $T \leqslant T_1$ (this limitation is a natural one from physical considerations), then the function $T(R)$ decreases monotonically. Introduce the dimensionless quantities

$$x = \frac{R_2 - R}{R_2 - R_1}, \qquad h = \frac{R_2 - R_1}{R_2}, \qquad \vartheta = \frac{T - T_2}{T_1 - T_2}, \qquad w = \frac{W}{W_1}$$
$$\alpha(\vartheta) = \frac{\lambda(T)}{\lambda_2}, \qquad \beta(\vartheta) = \frac{\mu(T)}{\mu_2} \qquad (\lambda_2 = \lambda_{T=T_2},\ \mu_2 = \mu_{T=T_2}) \tag{1.8}$$

Then equations (1.3) and (1.5), and the boundary conditions (1.6), (1.7) become

$$\frac{d}{dx}\frac{w}{1-hx} - \frac{a}{\beta(\vartheta)(1-hx)^3} = 0 \tag{1.9}$$

$$\frac{d}{dx}\left[\alpha(\vartheta)(1-hx)\frac{d\vartheta}{dx}\right] + \varkappa\frac{a^2}{\beta(\vartheta)(1-hx)^3} = 0 \tag{1.10}$$

$$\vartheta(0) = w(0) = 0, \qquad \vartheta(1) = w(1) = 1 \qquad (0 \leqslant x \leqslant 1,\ h < 1) \tag{1.11}$$

where a is an unknown constant, and

$$\varkappa = \frac{W_1^2 \mu_2}{(T_1 - T_2)\lambda_2} \tag{1.12}$$

In general, for arbitrary $\alpha(\theta)$ and $\beta(\theta)$, the system of equations (1.9), (1.10) cannot be solved in quadratures. The basic purpose of this work is to construct approximate formulas, using the small parameter method, for as general a situation as possible, taking as a point of departure certain special cases where the solution can be obtained by quadratures.

2. The case of small κ. In certain cases the constant κ is small; this happens, for example, when the medium is air or water, while the speed W_1 is not too large, and the temperature difference, $T_1 - T_2$, is not too small.

For example, if $W_1 = 20$ m sec^{-1}, $T_1 = 80^\circ$C, $T_2 = 30^\circ$C, then formula (1.12) gives for air, water, and lubricating oil respectively,

$$\varkappa = 5.79 \cdot 10^{-3}, \qquad \varkappa = 1.04 \cdot 10^{-2}, \qquad \varkappa = 8.9$$

Here the values of λ and μ were taken from [4] (p. 22), and the value of κ for lubricating oil is shown to emphasize that κ need not be small.

Assuming κ is small, one can use the expansion

$$\vartheta = \vartheta_0 + \varkappa\vartheta_1 + \varkappa^2\vartheta_2 + \ldots, \qquad w = w_0 + \varkappa w_1 + \varkappa^2 w_2 + \ldots,$$
$$a = a_0 + \varkappa a_1 + \varkappa^2 a_2 + \ldots \tag{2.1}$$

The terms in the series are determined from the equations obtained by substituting the series in (1.9), (1.10), and equating coefficients of like powers of κ. Thus, the terms of lowest order satisfy the same equations (1.9)-(1.11), but with $\kappa = 0$. In this case it is easy to integrate the equations. From (1.10) and the boundary values for θ_0 we get

$$\int_0^{\vartheta_0} \alpha(\vartheta)\, d\vartheta = \frac{\ln(1-hx)}{\ln(1-h)} \int_0^1 \alpha(\vartheta)\, d\vartheta \tag{2.2}$$

From this, one determines the function $\theta_0(x)$. Substituting this result in (1.9) and taking into account the boundary conditions for w_0 we get

$$w_0(x) = a_0(1-hx)\int_0^x \frac{dx}{\beta[\vartheta_0(x)]\,(1-hx)^3}, \quad a_0 = \left[(1-h)\int_0^1 \frac{dx}{\beta[\vartheta_0(x)]\,(1-hx)^3}\right]^{-1} \tag{2.3}$$

Equations determining the terms of first order are obtained by equating the coefficients of κ to the first power. In doing this, of course, one must use expressions of the form

$$\alpha(\vartheta) = \alpha(\vartheta_0) + \varkappa\vartheta_1\alpha'(\vartheta_0) + \varkappa^2\left[\vartheta_2\alpha'(\vartheta_0) + \frac{\vartheta_1^2}{2}\alpha''(\vartheta_0)\right] + \ldots$$

Taking this remark into account, the aforementioned equations take the form:

$$\frac{d}{dx}\frac{w_1}{1-hx}-\frac{a_1}{\beta(\vartheta_0)(1-hx)^3}+\frac{a_0\vartheta_1\beta'(\vartheta_0)}{[\beta(\vartheta_0)]^2(1-hx)^3}=0$$

$$\frac{d}{dx}\left[\alpha(\vartheta_0)(1-hx)\frac{d\vartheta_1}{dx}+\vartheta_1\alpha'(\vartheta_0)(1-hx)\frac{d\vartheta_0}{dx}\right]+\frac{a_0^2}{\beta(\vartheta_0)(1-hx)^3}=0$$

$$\vartheta_1(0)=w_1(0)=\vartheta_1(1)=w_1(1)=0$$

where the functions $\theta_0(x)$, $w_0(x)$ and the constant a_0 have already been evaluated. On integrating we get

$$\vartheta_1=\frac{a_0^2}{\alpha(\vartheta_0)}\left(\frac{\ln(1-hx)}{\ln(1-h)}\int_0^1\frac{G_1(x)\,dx}{1-hx}-\int_0^x\frac{G_1(x)\,dx}{1-hx}\right) \tag{2.4}$$

$$w_1=(1-hx)\left[a_1G_1(x)-a_0G_2(x)\right] \qquad \left(a_1=a_0\frac{G_2(1)}{G_1(1)}\right) \tag{2.5}$$

$$G_1(x)=\int_0^x\frac{dx}{\beta(\vartheta_0)(1-hx)^3}, \qquad G_2(x)=\int_0^x\frac{\vartheta_1\beta'(\vartheta_0)\,dx}{[\beta(\vartheta_0)]^2(1-hx)^3} \tag{2.6}$$

In an analogous way, if it is desired, one can find the terms of second and higher orders in κ.

Let us examine the example of a liquid with small κ (water, in particular). Here we can take

$$\alpha(\vartheta)\equiv 1, \qquad \beta(\vartheta)=\frac{1}{1+\beta^\circ(\vartheta)}, \qquad \beta^\circ=\frac{\mu_2-\mu_1}{\mu_1}$$

This case was examined in [2] in general form; as it turned out, the solution can be expressed in terms of Bessel Functions (let us note that for large values of the argument it is convenient to use the asymptotic representation of these functions). If κ is small, one can obtain an approximate formula which uses only elementary functions. In fact, calculations from (2.2) and (2.3) yield

$$\vartheta_0=\frac{\ln(1-hx)}{\ln(1-h)}$$

$$w_0=\frac{a_0}{1-hx}\left[\frac{x(2-hx)}{2}+\beta^\circ\,\frac{2\ln(1-hx)+hx(2-hx)}{4h\ln(1-h)}\right]$$

$$a_0=(1-h)\left[\frac{2-h}{2}+\beta^\circ\,\frac{2\ln(1-h)+h(2-h)}{4h\ln(1-h)}\right]^{-1}$$

Corrections for κ are computed from (2.4) and (2.5), and in the process all the integrals can be evaluated and yield functions similar to those in the expression for $w_0(x)$, although the results are quite cumbersome. Let us limit ourselves to the case of a small dimensionless gap between stator and rotor (i.e. small h), where one can retain terms of first order only in h and κ. The computation gives

$$\vartheta_0 = x - h\frac{x(1-x)}{2} + \ldots, \qquad \vartheta_1 = \frac{2x(1-x)}{3(2+\beta^\circ)^2}(3 + \beta^\circ + \beta^\circ x) + \ldots$$

$$w_0 = \frac{x(2+\beta^\circ x)}{2+\beta^\circ} - h\frac{x(1-x)}{3(2+\beta^\circ)^2}(1 + 10\beta^\circ + 16\beta^\circ x + 8\beta^{\circ 2} x) + \ldots$$

$$w_1 = \frac{\beta^\circ x^2(1-x)}{3(2+\beta^\circ)^3}(4 + \beta^\circ + \beta^\circ x) + \ldots$$

Thus, for small h and κ for a liquid we obtain the approximate formulas (with an error of second order)

$$\vartheta \approx x - h\frac{x(1-x)}{2} + \kappa\frac{2x(1-x)}{3(2+\beta^\circ)^2}(3 + \beta^\circ + \beta^\circ x)$$

$$w \approx \frac{x(2+\beta^\circ x)}{2+\beta^\circ} - h\frac{x(1-x)}{3(2+\beta^\circ)^2}(\lambda + 10\beta^\circ + 16\beta^\circ x + 8\beta^{\circ 2} x) +$$
$$+ \kappa\frac{\beta^\circ x^2(1-x)}{3(2+\beta^\circ)^3}(4 + \beta^\circ + \beta^\circ x)$$

If the thermal conductivity (and with it the quantity a) cannot be considered constant, then to improve the result one can use $1/2(\lambda_1 + \lambda_2)$ instead of λ_2, or one can let $a(\theta) = 1 + a_0(\theta)$ with small a_0, and, expanding in powers of a_0, retain terms of first order only.

In conclusion, let us note conditions under which for an arbitrary medium (i.e. arbitrary $a(\theta)$ and $\beta(\theta)$ it is permissible to discard the terms in κ in solving equations (1.9) and (1.10), i.e. to accept θ_0 and w_0, given by (2.2) and (2.3), as the exact solution. In order to do this, let us integrate equations (1.9) and (1.10), and in these integrals replace the variable quantities by their mean values (these will be denoted by stars, and they can take on different values in different formulas). In this way we will obtain successively

$$\frac{w}{1-hx} = \frac{a}{\beta^*}\int_0^x \frac{dx}{(1-hx)^3} = \frac{a(2x-hx^2)}{2\beta^*(1-hx)^2}, \qquad a = \frac{2\beta^*(1-h)}{2h} \tag{2.7}$$

$$\alpha(\vartheta)(1-hx)\frac{d\vartheta}{dx} = C - \kappa\frac{a^2(2x-hx^2)}{2\beta^*(1-hx)^2}$$

$$\int_0^\vartheta \alpha(\vartheta)\,d\vartheta = -\frac{C}{h}\ln(1-hx) - \frac{\kappa a^2}{2\beta^*}\left[\frac{1}{2h^2}\left(\frac{1}{(1-hx)^2} - 1\right) + \frac{1}{h^2}\ln(1-hx)\right]$$

$$C = -\frac{h}{\ln(1-h)}\int_0^1 \alpha(\vartheta)\,d\vartheta - \frac{h}{\ln(1-h)}\frac{\kappa a^2}{2\beta^*}\left[\frac{1}{2h^2}\left(\frac{1}{(1-h)^2} - 1\right) + \frac{1}{h^2}\ln(1-h)\right]$$

$$\int_0^\vartheta \alpha(\vartheta)\,d\vartheta = \frac{\ln(1-hx)}{\ln(1-h)}\int_0^1 \alpha(\vartheta)\,d\vartheta + \frac{\kappa a^2}{2}F(x)$$

Here

$$F(x) = \frac{\ln(1-hx)}{\ln(1-h)}\frac{1}{\beta^*}\left[\frac{1}{2h^2}\left(\frac{1}{(1-h)^2}-1\right)+\frac{1}{h^2}\ln(1-h)\right]-$$
$$-\frac{1}{\beta^*}\left[\frac{1}{2h^2}\left(\frac{1}{(1-hx)^2}-1\right)+\frac{1}{h^2}\ln(1-hx)\right] \qquad (2.8)$$

If in the last expression we set $\kappa = 0$, it will go over into equation (2.2), with θ changing by $\Delta\theta$. From the mean value theorem it follows that

$$|\Delta\vartheta| = \frac{\varkappa a^2}{2\alpha^*}|F(x)|$$

Since both of the terms in expression (2.8) for $F(x)$ are positive and (if we do not consider β^*) monotone increasing,

$$|\Delta\vartheta| \leqslant \frac{\varkappa a^2}{2\alpha^*\beta^*}\left[\frac{1}{2h^2}\left(\frac{1}{(1-h)^2}-1\right)+\frac{1}{h^2}\ln(1-h)\right] \qquad (2.9)$$

Thus, discarding the term in κ in (1.10) leads to an error in $T(R)$, which, in absolute value, does not exceed

$$\frac{2h-h^2+2(1-h)^2\ln(1-h)}{(2-h)^2h^2}\,\frac{\mu_2\mu_{max}}{\mu_{min}\lambda_{min}}W_1^2 \qquad (2.10)$$

(obtained by transforming the right-hand side of (2.9)).

If this error is smaller than the accuracy required in determining T, then it is certainly permissible to discard the terms indicated above. Let us note that the expression (2.10) can be simplified in the case of a small gap: if we neglect terms of order h, it goes into

$$\frac{\mu_2\mu_{max}}{\lambda_{min}\mu_{min}}\,\frac{W_1^2}{2}$$

For example, if the gap contains air, and $0°C \leqslant T \leqslant 200°C$, taking the values for μ_{max} from [6] (p. 256), we find that expression (2.10) exceeds 1° only if

$$W_1 > \sqrt{\frac{5.76\cdot10^{-6}\cdot427}{2.64\cdot10^{-6}}\cdot 2} = 42.8\text{ m sec}^{-1}$$

The possibility of neglecting terms in κ in equation (2.10) really means that under the conditions under examination one can consider the cylinders stationary in computing temperatures.

3. The case of slowly varying λ and μ. For gases, in particular for air, the coefficients λ and μ vary slowly; the same situation prevails also for other media if the temperature difference, $T_1 - T_2$, is not too large. In this case one can put

$$\alpha(\vartheta) = 1 + \alpha_0\varphi(\vartheta), \qquad \beta(\vartheta) = 1 + \beta_0\psi(\vartheta)$$

where α_0 and β_0 are small. Such formulas include all the possible cases of $\alpha(\theta)$ and $\beta(\theta)$ which can occur. Equations (1.9), (1.10) can be rewritten in the form

$$(1+\beta_0\psi(\vartheta))\frac{d}{dx}\left(\frac{w}{1-hx}\right)-\frac{a}{(1-hx)^3}=0 \tag{3.1}$$

$$(1+\beta_0\psi(\vartheta))\frac{d}{dx}\left[(1+\alpha_0\varphi(\vartheta))(1-hx)\frac{d\vartheta}{dx}\right]+\frac{\varkappa a^2}{(1-hx)^3}=0 \tag{3.2}$$

We can look for solutions in powers of α_0, β_0:

$$\vartheta=\vartheta_0+\alpha_0\vartheta_1+\beta_0\vartheta_2+\ldots,\qquad w=w_0+\alpha_0w_1+\beta_0w_2+\ldots,$$
$$a=a_0+\alpha_0a_1+\beta_0a_2+\ldots$$

For the principal terms we get equations (3.1), (3.2), in which $\alpha_0=\beta_0=0$, i.e. we get the case of constant λ and μ. Then the equation (with boundary conditions (1.11)) can be easily integrated (see [4], p. 379, for example); we get

$$\vartheta_0=\frac{\ln(1-hx)}{\ln(1-h)}+\frac{\varkappa}{h(2-h)^2}\left[\frac{(2-h)\ln(1-hx)}{\ln(1-h)}-\frac{(1-h)^2x(2-hx)}{(1-hx)^2}\right] \tag{3.3}$$

$$w_0=\frac{(1-h)\,x\,(2-hx)}{(2-h)(1-hx)},\qquad a_0=\frac{2(1-h)}{2-h} \tag{3.4}$$

In order to find the terms of first order in α_0, β_0, we equate the coefficients of α_0 and β_0 in (3.1) and (3.2); this gives

$$\frac{d}{dx}\left(\frac{w_1}{1-hx}\right)-\frac{a_1}{(1-hx)^3}=0$$

$$\frac{d}{dx}\left(\frac{w_2}{1-hx}\right)+\psi(\vartheta_0)\frac{d}{dx}\left(\frac{w_0}{1-hx}\right)-\frac{a_2}{(1-hx)^3}=0$$

$$\frac{d}{dx}\left[(1-hx)\frac{d\vartheta_1}{dx}\right]+\frac{d}{dx}\left[\varphi(\vartheta_0)(1-hx)\frac{d\vartheta_0}{dx}\right]+\frac{2\varkappa a_0a_1}{(1-hx)^3}=0$$

$$\frac{d}{dx}\left[(1-hx)\frac{d\vartheta_2}{dx}\right]+\varphi(\vartheta_0)\frac{d}{dx}\left[(1-hx)\frac{d\vartheta_0}{dx}\right]+\frac{2\varkappa a_0a_2}{(1-hx)^3}=0$$

From this, taking into account the homogeneous boundary conditions for w_0 and θ_0, it is easy to find all the functions w_1, w_2, θ_1, θ_2, and the constants a_1, a_2 explicitly:

$$w_1=a_1\frac{x(2-hx)}{2(1-hx)},\qquad 0=a_1\frac{2-h}{2(1-h)}\ \text{ or }\ a_1=0,\qquad w_1=0$$

$$w_2=(1-hx)\left[a_2\frac{x(2-hx)}{2(1-hx)^2}-a_0G_3(x)\right],\qquad a_2=\frac{4(1-h)^3}{(2-h)^2}G_3(1),$$

or

$$w_2 = \frac{2(1-h)(1-hx)}{(2-h)}\left[\frac{(1-h)^2 x(2-hx)}{(2-h)(1-hx)^2} G_3(1) - G_3(x)\right]$$

$$\vartheta_1 = \frac{\ln(1-hx)}{\ln(1-h)} G_4(1) - G_4(x)$$

$$\vartheta_2 = \frac{4\varkappa(1-h)^2}{(2-h)^2}\left\{\left[\int_0^x \frac{G_5(x)\,dx}{1-hx} - \frac{\ln(1-hx)}{\ln(1-h)}\int_0^1 \frac{G_5(x)\,dx}{1-hx}\right] - \right.$$

$$\left. - \frac{(1-h)^2}{h(2-h)}\left[\frac{x(2-hx)}{(1-hx)^2} - \frac{\ln(1-hx)}{\ln(1-h)}\,\frac{2-h}{(1-h)^3}\right] G_3(1)\right.$$

$$G_3(x) = \int_0^x \frac{\psi(\vartheta_0)\,dx}{(1-hx)^3}, \qquad G_4(x) = \int_0^x \varphi(\vartheta_0)\frac{d\vartheta_0}{dx}\,dx, \qquad G_5(x) = \int_0^x \frac{G(\vartheta_0)dx}{(1-hx)^3}$$

In a similar way we can express terms of second and higher order.

As an example, let us examine the case where $\lambda(T)$ and $\mu(T)$ are linear, and κ is quite small (in particular, this takes place for an air gap). Then, one can put

$$\alpha(\vartheta) = 1 + \alpha_0\vartheta; \qquad \beta(\vartheta) = 1 + \beta_0\vartheta, \quad \text{or} \quad \varphi(\vartheta) \equiv \psi(\vartheta) \equiv \vartheta$$

where the constants a_0 and β_0 are obtained from

$$\alpha_0 = \frac{\lambda_1 - \lambda_2}{\lambda_2}, \qquad \beta_0 = \frac{\mu_1 - \mu_2}{\mu_2}$$

(Let us note that if, as frequently happens, the Prandtl number is independent of the temperature, we must set $a_0 = \beta_0$, which simplifies somewhat the formula obtained.)

Let us restrict ourselves to terms of second order with respect to a_0 and β_0, and to terms of first order with respect to κ. In doing this one can use the method just described; however, it is simpler to bring in also formulas (2.2) and (2.3) for the group of terms which do not depend on κ.

From (2.2) we obtain for this group of terms the equation

$$\vartheta_0 + \frac{\alpha_0}{2}\vartheta_0^{\,2} = \left(1 + \frac{\alpha_0}{2}\right) y \qquad \left(\frac{\ln(1-hx)}{\ln(1-h)} = y,\ 0 \leqslant y \leqslant 1\right)$$

Consequently,

$$\vartheta_0 = y + \alpha_0\frac{y(1-y)}{2} - \alpha_0^{\,2}\frac{y^2(1-y)}{2} + \ldots$$

Substitution of this result in (2.3) yields, after some computations, an expression for w_0; retaining only the terms of first order, we get

$$w_0 = \frac{(1-h)\,x\,(2-hx)}{(2-h)\,(1-hx)} + \beta_0 \frac{(1-h)}{(2-h)\,h\,(1-hx)}\left(x\,\frac{2-hx}{2-h} - y\right) + \ldots \qquad (3.5)$$

(Let us note that here θ_0, w_0 have the same meaning as in Section 2, and not as in the beginning of the present paragraph.) As for the correction in κ, we obtain it from formulas (3.3), (3.4). In particular, from (3.4) it is evident that in the expression for w, the term which is of first order in κ and independent of a_0 and β_0 is absent. Combining these results, we obtain an approximate expression for θ:

$$\vartheta \approx y + \alpha_0 \frac{y\,(1-y)}{2} - \alpha_0^{\,2}\,\frac{y^2\,(1-y)}{2} + \frac{\varkappa}{h\,(2-h)^2}\left[(2-h)\,y - \frac{(1-h)^2\,x\,(2-hx)}{(1-hx)^2}\right]$$

The term in $a_0^{\,2}$ in this formula does not exceed $(2/27)\,a_0^{\,2}$ and therefore, gives in the expression for $T(R)$ a term not exceeding

$$\frac{2}{27}\left(\frac{\lambda_1-\lambda_2}{\lambda_2}\right)^2(T_1-T_2)$$

This term plays a role only for a very large temperature drop; for example, if $0° \leqslant T \leqslant 200°$C, then for $T_1 - T_2 < 125°$ it is less than $1°$. The term in κ begins to play a role only for sufficiently large W_1; for example, one can calculate that for a small dimensionless gap h, for $0° \leqslant T \leqslant 200°$C, it is of the order of $1°$ only for $W_1 > 70$ m sec^{-1}. If the temperature drop, $T_1 - T_2$, and the velocity, W_1, are not too large, one can use the convenient formula

$$\vartheta \approx y + \alpha_0 \frac{y\,(1-y)}{2}$$

with the expression for w being given by (3.5).

If the dimensionless gap, h, is small, one can further expand with respect to h and retain terms of first order in h only. If, furthermore, we retain terms of no higher than second order in a_0, β_0, and first order in κ, then we obtain, after some calculations,

$$\vartheta \approx x + \alpha_0 \frac{x\,(1-x)}{2} - \alpha_0^{\,2}\,\frac{x^2\,(1-x)}{2} + \varkappa\,\frac{x\,(1-x)}{2} - h\,\frac{x\,(1-x)}{2}$$

$$w \approx x + \beta_0\,\frac{x\,(1-x)}{2} - \alpha_0\beta_0\,\frac{x\,(1-x)\,(1-2x)}{12} - \beta_0^{\,2}\,\frac{x\,(1-x)\,(1+4x)}{12} - h\,\frac{x\,(1-x)}{2}$$

If it is possible to discard terms in $a_0^{\,2}$, $a_0\beta_0$, $\beta_0^{\,2}$, κ and h, the formulas become extremely simple.

4. The case of a gap small compared to the radius. In this case we can use the expansions

$$\vartheta = \vartheta_0 + h\vartheta_1 + h^2\vartheta_2 + \ldots, \qquad w = w_0 + hw_1 + h^2w_2 + \ldots,$$

$$a = a_0 + ha_1 + h^2a_2 + \ldots$$

The equations for the principal terms have the same form as the equations (1.9)-(1.11) with $h = 0$; this corresponds to the case where the cylinders degenerate into parallel planes. Under this condition the equations have been integrated in a general form in [5]. The equations for the terms of first order then become

$$\frac{dw_1}{dx} + \frac{d}{dx}(xw_0) - \frac{a_1}{\beta(\vartheta_0)} + \frac{a_0\vartheta_1\beta'(\vartheta_0)}{[\beta(\vartheta_0)]^2} - \frac{3a_0x}{\beta(\vartheta_0)} = 0$$

$$\frac{d}{dx}\left[\vartheta_1\alpha'(\vartheta_0)\frac{d\vartheta_0}{dx} - x\alpha(\vartheta_0)\frac{d\vartheta_0}{dx} + \alpha(\vartheta_0)\frac{d\vartheta_1}{dx}\right] + $$
$$+ \frac{2\varkappa a_0a_1}{\beta(\vartheta_0)} - \frac{\varkappa a_0^2\vartheta_1\beta'(\vartheta_0)}{[\beta(\vartheta_0)]^2} + \frac{3\varkappa a_0^2x}{\beta(\vartheta_0)} = 0$$

This system of equations, although linear, is not in general solvable. However, it can be solved in certain special cases. This is the case for liquids (for all κ). Then the system of equations for the principal terms has constant coefficients, and can be solved by using trigonometric functions (for example, [4], p. 382). In this case the equation for θ_1 also has constant coefficients, and after θ_1 is found, the function, w_1, can be found by a simple integration.

BIBLIOGRAPHY

1. Wendt, F., Turbulente Strömungen zwischen zwei rotierenden konaxialen Zylindern. *Ing.-Arch.*, No. 6, 577-595, 1933.

2. Gorazdovskii, T.Ia. and Rigerer, S.A., Dvizhenie n'iutonovskoi zhidkosti mezhdu vrashchaiushchimisia koaksial'nymi tsilindrami pri nalichii vautrenykh teplovykh protsessov, vliiaiushchie na viazkie svoistva (The motion of a Newtonian fluid between rotating coaxial cylinders, in the presence of internal thermal processes which affect the viscous properties). *Zh. tekh. fiz.* Vol. 26, No. 7, pp. 1532-1541, 1956.

3. Stepaniants, L.G., Nekotorye sluchai dvizhenia szhimaemogo viazkogo gaza (Certain cases of the motion of a compressible viscous gas). Trud. Leningrad politekhn. in-ta im. M.I. Kalinina. *Energomashinostroenie, tekhnicheskaia gidromakhanika*, No. 5, 111-128, 1953.

4. Targ, S.G., *Osnovnye zadachi teorii laminarnykh techenii (Fundamental Problems in the Theory of Laminar Flow)*. GTTI, 1951.

5. Borisenko, O.I., Shvidkisne ta temperaturne pole v shari ridini mizh dvoma plastinkami pri vidnosnomu rusi odnei paralel'no drugii (Velocity and temperature fields across the gap between two plates moving parallel to each other). *Prikl. Mekh.* Vol. 2, pp. 425-437, 1956.

6. Schlichting, G., *Teoria pogranichnogo sloia* (*Boundary Layer Theory*). Izd-vo inostr. lit., Moscow, 1956.

Translated by M.V.